1. Suppose $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Let $\mathbf{b} = (3, -4, 0, 5)$. Solve $A\mathbf{x} = \mathbf{b}$ by finding solutions to two triangular systems.

(b) Find inverses (or explain why they don't exist) for L and U.

(c) Write an inverse for A in terms of L^{-1} and U^{-1}. Simplify your expression to write A^{-1} as a single matrix.

(d) Find a solution to $A\mathbf{x} = (-1, 4, 0, 0)$ using A^{-1}.

(e) Find an LU decomposition for A^T.

Solution:

(a) We write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 5 \end{pmatrix}$$

We first use forward substitution to solve the lower triangular system $L\mathbf{c} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 5 \end{pmatrix},$$

which has solution $\mathbf{c} = (c_1, c_2, c_3, c_4) = (3, -4, 0, 17)$.
Next we use back substitution to solve the upper triangular system $U\mathbf{x} = \mathbf{c}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 17 \end{pmatrix}$$

which has solution $\mathbf{x} = (x_1, x_2, x_3, x_4) = (3, -2, 17, 17)$.

(b) By using Gauss-Jordan with one row operation, or just noticing that L is an elimination matrix, we get

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}$$

We use Gauss-Jordan to find the inverse of U is $U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$[U|I] = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Multiply Row 2 by $1/2$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Add Row 4 to Row 3:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

(c)

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{pmatrix}$$

(d) A solution to $A\mathbf{x} = (-1, 4, 0, 0)$ is given by $\mathbf{x} = A^{-1}(-1, 4, 0, 0)$:

$$\mathbf{x} = A^{-1}(-1, 4, 0, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -12 \\ -12 \end{pmatrix}$$

(e)

$$A^T = (LU)^T = U^T L^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Let

$$A = \begin{pmatrix} 1 & 3 & 6 & -4 \\ 0 & 0 & -3 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 6 & -4 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & -3 & 4 \end{pmatrix}$$

(a) Find the RREF for A.
(b) Find the RREF for B.
(c) Find the rank of A, and the dimensions of $C(A)$, $N(A)$, $C(A^T)$, and $N(A^T)$.
(d) Find the rank of B, and the dimensions of $C(B)$, $N(B)$, $C(B^T)$, and $N(B^T)$.
(e) If $\mathbf{b} \in \mathbb{R}^2$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have? Explain how you know.
(f) Is $C(A^T) \subseteq C(B^T)$, $C(A^T) = C(B^T)$, $C(B^T) \subseteq C(A^T)$, or none of these? Explain briefly.
(g) Is $N(A) \subseteq N(B)$, $N(A) = N(B)$, $N(B) \subseteq N(A)$, or none of these? Explain briefly.

Solution:

(a) The RREF for A is obtained by adding twice Row 2 to Row 1, then multiplying Row 2 by $-1/3$. It is $\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

(b) The RREF for B is obtained by adding twice Row 2 to Row 1, subtracting Row 2 from Row 3, then multiplying Row 2 by $-1/3$, to get $\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then adding Row 3 to Row 2 and subtracting twice Row 3 from Row 1 we get the RREF: $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) The rank of A is 2, and
- $\dim C(A) = r = 2$
- $\dim N(A) = n - r = 4 - 2 = 2$
- $\dim C(A^T) = r = 2$
- $\dim N(A^T) = m - r = 2 - 2 = 0$

(d) The rank of B is 3, and
- $\dim C(B) = r = 3$
- $\dim N(B) = n - r = 4 - 3 = 1$
- $\dim C(B^T) = r = 3$
- $\dim N(B^T) = m - r = 3 - 3 = 0$

(e) Since A has full row rank (a solution exists), but not full column rank (two-dimensional null space), $A\mathbf{x} = \mathbf{b}$ always has an infinite number of solutions.

(f) Since the rows of A are all rows of B, and the third row of B is not in the span of the rows of A (it is a pivot row), $C(A^T) \subseteq C(B^T)$.

(g) Since the rows of A are rows of B, if $B\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$. However $\dim N(A) = 2$ while $\dim N(B) = 1$, so there are elements of $N(A)$ that are not in $N(B)$, so $N(B) \subseteq N(A)$.

3. Let $\mathbf{v} = (1, 0, 1, 1)$ and $\mathbf{w} = (1, 1, 0, 0)$ be vectors in $\mathbb{R}^4$. Let $V = \text{span}\{\mathbf{v}\}$ and $U = \text{span}\{\mathbf{v}, \mathbf{w}\}$.

   (a) Define what it means for a set of vectors to be linearly independent.

   (b) Prove or disprove using the definition of linear independence: The set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

   (c) What is the dimension of U? Explain how you know.

   (d) Prove or disprove: The vector $(0, 1, 0, 0)$ is in the orthogonal complement of V.

   (e) Prove or disprove: The vector $(0, 1, 0, 0)$ is in the orthogonal complement of U.

   (f) Prove that the orthogonal complement of U is closed under scalar multiplication.

   (g) Is $U \subseteq V$, $U = V$, $V \subseteq U$, or none of these? Explain.

   (h) Is $U^\perp \subseteq V^\perp$, $U^\perp = V^\perp$, $V^\perp \subseteq U^\perp$, or none of these? Explain.

   (i) Suppose P is the matrix such that $P\mathbf{b}$ is the projection of $\mathbf{b}$ onto U. Find a matrix A such that $P = A(A^TA)^{-1}A^T$. How many rows and columns does P have?

Solution:

   (a) A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent if the only linear combination $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ equal to $\mathbf{0}$ is the trivial linear combination where $a_1 = \cdots = a_n = 0$.

   (b) The set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. Note $a_1\mathbf{v}+a_2\mathbf{w} = (a_1+a_2, a_2, a_1, a_1)$. If this linear combination is equal to $(0, 0, 0, 0)$, then the second component implies that $a_2 = 0$, and the third component implies $a_1 = 0$. Thus the only linear combination equal to $\mathbf{0}$ is the trivial one.

   [Note this is equivalent to showing that the matrix $(\mathbf{v} \mid \mathbf{w}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$ has a trivial nullspace (full column rank).]

   (c) The dimension of U is 2. This is because the vectors $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent and their span is U. Hence they form a basis for U, and the dimension of a vector space is the number of vectors in a basis for that space.

   (d) This is true. An arbitrary element of V has the form $c\mathbf{v} = (c, 0, c, c)$, where c is a scalar. Since $(0, 1, 0, 0)^T(c, 0, c, c) = 0$, the vector $(0, 1, 0, 0)$ is orthogonal to all vectors in V, so it is in $V^\perp$.

   (e) This is false. Since $(0, 1, 0, 0)^T(1, 1, 0, 0) = 1 \neq 0$, $(0, 1, 0, 0)$ is not orthogonal to $(1, 1, 0, 0)$. Since $(1, 1, 0, 0) \in U$, $(0, 1, 0, 0)$ is not orthogonal to all vectors in U, hence is not in the orthogonal complement.

   (f) The orthogonal complement $U^\perp$ is closed under scalar multiplication. Suppose a vector $\mathbf{x}$ is in $U^\perp$, and consider $c\mathbf{x}$ where c is an arbitrary scalar. Let $\mathbf{y} \in U$. Then since $\mathbf{x}^T\mathbf{y} = 0$,

$$(c\mathbf{x})^T\mathbf{y} = c(\mathbf{x}^T\mathbf{y}) = c(0) = 0$$

Thus $c\mathbf{x} \in U^\perp$, and $U^\perp$ is closed under scalar multiplication.

(g) Since the basis $\{\mathbf{v}\}$ for V is a strict subset of the basis $\{\mathbf{v}, \mathbf{w}\}$ for U, $V \subseteq U$. (V is a 1-dimensional subspace of the 2-dimensional vector space U)

(h) Since $V \subseteq U$, every vector in $U^\perp$ must be in $V^\perp$. However there are vectors like $(0, 1, 0, 0)$ in $V^\perp$ that are not in $U^\perp$, so $U^\perp \subseteq V^\perp$.

(i) The matrix A can be any matrix whose columns form a basis for U, e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$. In any case, $P = A(A^TA)^{-1}A^T$ will have four rows and four columns.